The 32nd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1st May 2025
Category II

Problem 1 *Let $x_0 = a$, $x_1 = b$, $x_2 = c$ for given numbers $a, b, c \in \mathbb{R}$, and let $x_{n+2} = \dfrac{x_n + x_{n-1}}{2}$ for $n \geq 1$. Show that the sequence $(x_n)_{n=0}^{\infty}$ converges, and find its limit.*

[Marcin J. Zygmunt / University of Silesia in Katowice]

Solution We have

$$\begin{aligned} x_{n+4} - x_{n+3} &= \frac{x_{n+2} + x_{n+1}}{2} - \frac{x_{n+1} + x_n}{2} \;=\; \frac{1}{2}(x_{n+2} - x_n) \;=\; \frac{1}{2}\left(\frac{x_n + x_{n-1}}{2} - x_n\right) \\ &= -\frac{1}{4}(x_n - x_{n-1}), \end{aligned}$$

so (x_n) is a Cauchy sequence (in $\mathbb{R}$), hence it converges.

Let now $y_n = x_{n+1} + x_n + \frac{1}{2}x_{n-1}$ for $n \geqslant 1$. We have

$$\begin{aligned} y_{n+1} &= x_{n+2} + x_{n+1} + \frac{1}{2}x_n \;=\; \frac{x_n + x_{n-1}}{2} + x_{n+1} + \frac{1}{2}x_n \;=\; x_{n+1} + x_n + \frac{1}{2}x_{n-1} \\ &= y_n \\ &\;\vdots \\ &= y_1 \;=\; x_2 + x_1 + \frac{1}{2}x_0 \;=\; c + b + \frac{1}{2}a, \end{aligned}$$

As we have

$$\lim_{n\to\infty} y_n = 2\frac{1}{2} \lim_{n\to\infty} x_n$$

we finally get

$$\lim_{n\to\infty} x_n = \frac{a + 2b + 2c}{5}.$$

□

Problem 2 *Let A, B be two $n \times n$ complex matrices of the same rank, and let $k \in \mathbb{N}$. Prove that $A^{k+1}B^k = A$ if and only if $B^{k+1}A^k = B$.* [Pirmyrat Gurbanov and Murat Chashemov / IUHD, Turkmenistan]

Solution Our statement is symmetric in A and B, so it is enough to prove the "only if" implication. We assume that $A^{k+1}B^k = A$ and we prove that $B^{k+1}A^k = B$.

We have $\ker B \subseteq \ker A^{k+1}B^k = \ker A$. But $\operatorname{rank} A = \operatorname{rank} B$, so we get $\ker B = \ker A$. On the other hand, we have

$$\operatorname{rank} A = \operatorname{rank} A^{k+1}B^k \leq \operatorname{rank} A^{k+1} \leq \operatorname{rank} A.$$

So we have $\operatorname{rank} A^{k+1} = \cdots = \operatorname{rank} A^2 = \operatorname{rank} A$. It is clear that $\ker A \subseteq \ker A^2$, so we have $\ker A = \ker A^2$. Now we claim that $\mathbb{C}^n = \ker A \oplus \operatorname{Im} A$. In fact, it is enough to prove that $\ker A \cap \operatorname{Im} A = 0$. If $x \in \ker A \cap \operatorname{Im} A$, then there exists $y \in \mathbb{C}^n$ such that $x = Ay$, so $A^2y = Ax = 0$, i.e., $y \in \ker A^2 = \ker A$. This gives $x = Ay = 0$. So we can choose a basis such that A is of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where A_1 is invertible. Since $\ker A = \ker B$, we may assume that B is of the following form under the same basis:

$$\begin{pmatrix} B_1 & 0 \\ B_3 & 0 \end{pmatrix}.$$

Finally, from $A^{k+1}B^k = A$ we see that $A_1^{k+1}B_1^k = A_1$. So we have $A_1^kB_1^k = I_r$ since A_1 is invertible. Now it is easy to see that $B^{k+1}A^k = B$. $\square$

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Problem 3 *Evaluate the integral*

$$\int_0^\infty \frac{\log(x+2)}{x^2+3x+2}\,\mathrm{d}x\,.$$

[Asen Bozhilov / Sofia University]

Solution Using change of variables $x \to x-1$ we obtain

$$\int_0^\infty \frac{\ln(x+2)}{(x+1)(x+2)}\,dx = \int_1^\infty \frac{\ln(x+1)}{x(x+1)}\,dx$$

Let us define $I(a) := \int_1^\infty \frac{\ln(1+ax)}{x(1+x)}\,dx$. We need to find $I(1)$. Clearly $I(0)=0$. Moreover

$$I'(a) = \int_1^\infty \frac{\partial}{\partial a}\frac{\ln(1+ax)}{x(1+x)}\,dx = \int_1^\infty \frac{1}{(1+x)(1+ax)}\,dx = \frac{\ln\left(\frac{2a}{a+1}\right)}{a-1}.$$

Hence

$$I(1) = \int_0^1 I'(a)\,da = \int_0^1 \frac{\ln\left(\frac{2a}{a+1}\right)}{a-1}\,da$$

It is well-known that $J := \int_0^1 \frac{\ln a}{a-1}\,da = \frac{\pi^2}{6}$.
Thus

$$\begin{aligned} I(1) =& \int_0^1 \frac{\ln\left(\frac{2a}{a+1}\right)}{a-1}\,da = J - \int_0^1 \frac{\ln\left(\frac{a+1}{2}\right)}{a-1}\,da = J - \int_0^1 \frac{\ln\left(1-\frac{a}{2}\right)}{a}\,da \\ =& J - \int_0^{1/2} \frac{\ln(1-a)}{a}\,da. \end{aligned}$$

Thus we are left with finding the last integral which we denote by H. Then, using integration by parts, we obtain

$$\begin{aligned} H =& \int_0^{1/2} \log(1-a)\,d\log a = -\log a \log(1-a)\Big|_{a=0}^{1/2} - \int_0^{1/2} \frac{\log a}{1-a}\,da \\ =& -\ln^2 2 + \int_{1/2}^1 \frac{\ln(1-a)}{a}\,da \end{aligned}$$

On the other hand $H + \int_{1/2}^1 \frac{\ln(1-a)}{a}\,da = -J$. The last two relations between H and $\int_{1/2}^1 \frac{\ln(1-a)}{a}\,da$ allow us to find $H = -\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2$ □

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Problem 4 *Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane, and suppose that $f\colon D \to D$ is a holomorphic function fulfilling the property $\lim\limits_{|z|\to 1} |f(z)| = 1$. Let the Taylor series of f be $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. Prove that $\sum\limits_{n=0}^{\infty} n|a_n|^2$ is equal to the number of zeros of f (counted with multiplicities).*

[Géza Kós / Eötvös Loránd University, Budapest]

Solution If f is constant then $|a_0| = |f| = 1$ and $a_1 = a_2 = \ldots = 0$, so the statement is trivial.

Suppose that f is not constant. By the argument principle, the function attains all values the same number of times. Let this number be K. In particular, K is the number of zeros.

Since $f(D)$ covers D K times, the area of the range with multiplicities is $K\pi$. On the other hand, that area can be expressed as

$$\begin{aligned}
K\pi &= \int_{|z|\in D} \left|f'(z)\right|^2 |\mathrm{d}z|^2 \\
&= \int_{|z|\in D} \left(\sum_{n=1}^{\infty} n a_n z^{n-1}\right)\left(\sum_{k=1}^{\infty} \overline{k a_k z^{k-1}}\right)|\mathrm{d}z|^2 \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} nk a_n \overline{a_k} \int_{r=0}^{1} \underbrace{\int_{t=0}^{2\pi} r^{n+k-2} e^{i(n-k)t} \cdot r \,\mathrm{d}t}_{0 \text{ if } n \neq k} \,\mathrm{d}r \\
&= \sum_{n=0}^{\infty} n^2 |a_n|^2 \int_{r=0}^{1}\int_{t=0}^{2\pi} r^{2n-1} \,\mathrm{d}t\,\mathrm{d}r = \sum_{n=0}^{\infty} n^2|a_n|^2 \cdot \frac{2\pi}{2n} = \pi \sum_{n=0}^{\infty} n \cdot |a_n|^2,
\end{aligned}$$

so

$$\sum_{n=0}^{\infty} n \cdot |a_n|^2 = K.$$

□