The 32^{nd} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1^{st} May 2025
Category I

Problem 1 *Let $a \geq 2$ be an integer. Prove that there exists a positive integer b with the following property: For each positive integer n, there is a prime number $p = p(a, b, n)$ such that $a^n + b$ is divisible by p but not divisible by p^2.* [Artūras Dubickas / Vilnius University]

Solution Assume first that $a = 2$. Select $b = 10$. Then, for $n = 1$, the number $a^n + b = 2^1 + 10 = 12$ is divisible by 3 but not by 3^2, while for $n \geq 2$ the number $2^n + 10$ is divisible by 2 but not by 2^2. Similarly, for $a = 2^m$, where $m \geq 2$, it is clear that for $b = 10$ and for each positive integer n the number $a^n + b = 2^{mn} + 10$ is divisible by 2 but not by 2^2.

It remains to consider the case when a has an odd prime factor. Let q be an odd prime factor of a. If $q \mid a$ but $q^2 \nmid a$, then, selecting $b = a$, we see that $a^n + b = a^n + a$ is divisible by q but not by q^2. (For $n = 1$ we have $a^n + a = 2a$, so this is also true.) If $q^2 \mid a$, then, choosing $b = q$, we see that for each $n \geq 1$ the number $a^n + b = a^n + q$ is divisible by q but not by q^2. $\square$

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Problem 2 *Find all real numbers $x > 1$ that satisfy the equality*

$$\left\lfloor \frac{n+1}{x} \right\rfloor = n - \left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{\lfloor \frac{n}{x} \rfloor}{x} \right\rfloor - \left\lfloor \frac{\lfloor \frac{\lfloor \frac{n}{x} \rfloor}{x} \rfloor}{x} \right\rfloor + \dots$$

for all positive integers n.

Here, $\lfloor t \rfloor$ is the integer part of t, i.e. the integer such that $\lfloor t \rfloor \le t < \lfloor t \rfloor + 1$.

[Marcin J. Zygmunt / University of Silesia in Katowice]

Solution The assumption that $x > 1$ makes the sum on the right-hand side of the equation finite. Putting now $\lfloor \frac{n}{x} \rfloor$ for n in the equation gives

$$\left\lfloor \frac{\lfloor \frac{n}{x} \rfloor + 1}{x} \right\rfloor = \left\lfloor \frac{n}{x} \right\rfloor - \left\lfloor \frac{\lfloor \frac{n}{x} \rfloor}{x} \right\rfloor + \left\lfloor \frac{\lfloor \frac{\lfloor \frac{n}{x} \rfloor}{x} \rfloor}{x} \right\rfloor - \dots,$$

which, added to the equation, gives

$$\left\lfloor \frac{n+1}{x} \right\rfloor + \left\lfloor \frac{\lfloor \frac{n}{x} \rfloor + 1}{x} \right\rfloor = n.$$

Dividing both sides by n and taking the limit as $n \to \infty$ gives

$$\frac{1}{x} + \frac{1}{x^2} = 1,$$

which has the only solution greater that 1 equal to $\varphi = \dfrac{1+\sqrt{5}}{2}$.

It remains only to show that for $x = \varphi$ the equality is satisfied for all positive integers n. So let $\frac{n}{\varphi} = \left\lfloor \frac{n}{\varphi} \right\rfloor + \varepsilon$, with $0 < \varepsilon < 1$. Now

$$\frac{n+1}{\varphi^2} = \frac{n - n\varphi + n\varphi - 1}{\varphi^2} = n - \left\lfloor \frac{n}{\varphi} \right\rfloor + \frac{1}{\varphi^2} - \varepsilon$$

and

$$\frac{\left\lfloor \frac{n}{\varphi} \right\rfloor + 1}{\varphi} = \frac{n}{\varphi^2} + \frac{1-\varepsilon}{\varphi} = n - \left\lfloor \frac{n}{\varphi} \right\rfloor - \varepsilon + \frac{1-\varepsilon}{\varphi}$$

as $\dfrac{1}{\varphi^2} = 1 - \dfrac{1}{\varphi}$. Since both $-\varepsilon + \dfrac{1-\varepsilon}{\varphi}, \dfrac{1}{\varphi^2} - \varepsilon \in (-1, 1)$ and

$$\frac{1}{\varphi^2} - \varepsilon = -\varepsilon\Big(1 + \frac{1}{\varphi}\Big) + \frac{1}{\varphi} = -\varepsilon\varphi = \Big(\frac{1}{\varphi^2} - \varepsilon\Big)\varphi,$$

so $-\varepsilon + \dfrac{1-\varepsilon}{\varphi}$ and $\dfrac{1}{\varphi^2} - \varepsilon$ are either both nonnegative or both negative, and we have the equality

$$\left\lfloor \frac{\left\lfloor \frac{n}{\varphi} \right\rfloor + 1}{\varphi} \right\rfloor = \left\lfloor \frac{n+1}{\varphi^2} \right\rfloor = \left\lfloor (n+1)\Big(1 - \frac{1}{\varphi}\Big) \right\rfloor = (n+1) + \left\lfloor -\frac{n+1}{\varphi} \right\rfloor = n - \left\lfloor \frac{n+1}{\varphi} \right\rfloor$$

as $\lfloor -\alpha \rfloor = -\lfloor \alpha \rfloor - 1$ for any noninteger α. Thus

$$\begin{aligned} \left\lfloor \frac{n+1}{\varphi} \right\rfloor &= n - \left\lfloor \frac{\lfloor \frac{n}{\varphi} \rfloor + 1}{\varphi} \right\rfloor = n - \left\lfloor \frac{n}{\varphi} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{\lfloor \frac{n}{\varphi} \rfloor}{\varphi} \right\rfloor + 1}{\varphi} \right\rfloor \\ &\ \ \vdots \\ &= n - \left\lfloor \frac{n}{\varphi} \right\rfloor + \left\lfloor \frac{\lfloor \frac{n}{\varphi} \rfloor}{\varphi} \right\rfloor - \left\lfloor \frac{\lfloor \frac{\lfloor \frac{n}{\varphi} \rfloor}{\varphi} \rfloor}{\varphi} \right\rfloor + \dots \end{aligned}$$

for any positive integer n. □

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Problem 3 *Let us call a sequence* $(b_1, b_2, \dots)$ *of positive integers* fast-growing *if* $b_{n+1} \geq b_n + 2$ *for all* $n \geq 1$. *Also, for a sequence* $a = \big(a(1), a(2), \dots\big)$ *of real numbers and a sequence* $b = (b_1, b_2, \dots)$ *of positive integers, let us denote*

$$S(a,b) = \sum_{n=1}^{\infty} \big| a(b_n) + a(b_n + 1) + \dots + a(b_{n+1} - 1) \big| .$$

1. *Do there exist two fast-growing sequences* $b = (b_1, b_2, \dots)$, $c = (c_1, c_2, \dots)$ *such that for every sequence* $a = \big(a(1), a(2), \dots\big)$, *if all the series*

$$\sum_{n=1}^{\infty} a(n)\,, \quad S(a,b) \quad \text{and} \quad S(a,c)$$

   are convergent, then the series $\sum\limits_{n=1}^{\infty} |a(n)|$ *is also convergent?*

2. *Do there exist three fast-growing sequences* $b = (b_1, b_2, \dots)$, $c = (c_1, c_2, \dots)$, $d = (d_1, d_2, \dots)$ *such that for every sequence* $a = \big(a(1), a(2), \dots\big)$, *if all the series*

$$S(a,b)\,, \quad S(a,c) \quad \text{and} \quad S(a,d)$$

   are convergent, then the series $\sum\limits_{n=1}^{\infty} |a(n)|$ *is also convergent?*

[Tomáš Bárta / Charles University, Prague]

Solution 2. Yes. Take $b_n = 2n - 1$, $c_n = 2n$, and $d_n = 3n - 2$. So, the convergent series are

$$|a_1 + a_2| + |a_3 + a_4| + |a_5 + a_6| + \dots$$
$$|a_2 + a_3| + |a_4 + a_5| + |a_6 + a_7| + \dots$$
$$|a_1 + a_2 + a_3| + |a_4 + a_5 + a_6| + \dots .$$

We have $|a_1| = |a_1 + a_2 + a_3 - (a_2 + a_3)| \leq |a_1 + a_2 + a_3| + |a_2 + a_3|$, similarly $|a_3| \leq |a_1 + a_2 + a_3| + |a_1 + a_2|$. Further, $|a_2| = |a_1 + a_2 - a_1| \leq |a_1 + a_2| + |a_1| \leq |a_1 + a_2| + |a_1 + a_2 + a_3| + |a_2 + a_3|$. The same holds for any triplet a_{3k+1}, a_{3k+2}, a_{3k+3}. Together we have

$$|a_{3k+1}| + |a_{3k+2}| + |a_{3k+3}| \leq 3(|a_{3k+1} + a_{3k+2}| + |a_{3k+2} + a_{3k+3}| + |a_{3k+1} + a_{3k+2} + a_{3k+3}|).$$

It follows that the sequence of partial sums $\sum_{n=1}^{N} |a_n|$ is bounded by $3(S_1 + S_2 + S_3)$, where S_1, S_2, S_3 are sums of the three given series respectively.

1. No. Case 1: there exists n_0 such that $b_{n+1} = b_n + 2$, $c_{n+1} = c_n + 2$ for all $n \geq n_0$. Then for $a_n = (-1)^n/n$ we have $|a(b_n) + a(b_n + 1)| = 0 = |a(c_n) + a(c_n + 1)|$ for all $n \geq n_0$, so both series $\sum |a(b_n) + \dots + a(b_{n+1} - 1)|$ and $\sum |a(c_n) + \dots + a(c_{n+1} - 1)|$ converge, also $\sum a_n$ is convergent, but $\sum |a_n| = \sum 1/n = +\infty$.

Case 2: Either (b_n) has infinitely many elements with $b_{n+1} \geq b_n + 3$ or (c_n) has this property. Let WLOG (b_n) has the property. Then for each n satisfying $b_{n+1} \geq b_n + 3$, the terms $a(b_n)$, $a(b_n + 1)$, $a(b_n + 2)$ are among the summands in $|a(b_n) + \dots + a(b_{n+1} - 1)|$, let us call it the three terms 'are in the same b-group'. However, either $a(b_n)$, $a(b_n + 1)$ are in the same c-group (if $b_n + 1 \neq c_k$ for all k) or $a(b_n + 1)$, $a(b_n + 2)$ are in the same c-group (if $b_n + 1 = c_k$ for some k). Hence, we have infinitely many couples $(a(i_k), a(i_k + 1))$, $k = 1, 2, \dots$ with each couple being in the same b-group and also in the same c-group. Now, it is sufficient to put $a(i_k) = 1/k$, $a(i_k + 1) = -1/k$ (we can assume $i_{k+1} > i_k + 1$) and all the remaining $a(n) = 0$. Then all the summands in $\sum |a(b_n) + \dots + a(b_{n+1} - 1)|$ and $\sum |a(c_n) + \dots + a(c_{n+1} - 1)|$ are zero and obviously, $\sum a_n$ is convergent (its partial sums are either 0 or $1/k$), but $\sum |a_n| = +\infty$ since it contains all $1/n$ and has non-negative terms. □

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Problem 4 *Let $n \geq 2$ be an integer, and let A be an $n \times n$ real matrix with minimal polynomial $x^n + x - 1$. Show that*

$$\operatorname{tr}\big((nA^{n-1} + I)^{-1} A^{n-2}\big) = 0\,.$$

Here, $\operatorname{tr}(B) = \sum_{i=1}^{n} b_{ii}$ *denotes the trace of the matrix* $B = (b_{ij})_{i,j=1}^{n}$.

[Marcin J. Zygmunt / University of Silesia in Katowice]

Solution The eigenvalues $\lambda_1, \dots \lambda_n$ of the matrix A satisfy equality $\lambda^n + \lambda = 1$, i.e. they are zeroes of the polynomial $\alpha(z) = z^n + z - 1$. The trace in question is equal to

$$\sum_{k=1}^{n} \frac{\lambda_k^{n-2}}{n\lambda_k^{n-1} + 1}\,,$$

where λ_k are the eigenvalues of A. We need the following two lemmas

Lemma 1. *Let $P(z) = a \prod_{k=1}^{n} (z - a_k)$. Then $\dfrac{P'(z)}{P(z)} = \sum_{k=1}^{n} \dfrac{1}{z - a_k}$ for $z \neq a_1, \dots, a_n$.*

Proof We have $P'(z) = a \sum_{k=1}^{n} \prod_{j \neq k} (z - a_j)$. But

$$\frac{\prod_{j \neq k} (z - a_j)}{\prod_{j=1}^{n} (z - a_j)} = \frac{1}{z - a_k}$$

for fixed $1 \leqslant k \leqslant n$. Hence

$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{\prod_{j \neq k} (z - a_j)}{\prod_{j=1}^{n} (z - a_j)} = \sum_{k=1}^{n} \frac{1}{z - a_k}\,.$$

□

Lemma 2. *Let $z_1, \dots, z_n$ be the roots of polynomial P of n-th degree, which are all distinct. Then*

$$\sum_{k=1}^{n} \frac{P''(z_k)}{P'(z_k)} = 0\,.$$

Proof Let $w_1, \dots, w_{n-1}$ be roots of the derivative P'. Since all roots of P are distinct, the numbers $w_1, \dots, w_{n-1}$ are all different from $z_1, \dots, z_n$, or equivalently $P'(z_j) \neq 0$ for all $j = 1, \dots, n$. Now Lemma 1. applied to the derivative P' gives $\dfrac{P''(z)}{P'(z)} = \sum_{j=1}^{n-1} \dfrac{1}{z - w_j}$ so

$$\sum_{k=1}^{n} \frac{P''(z_k)}{P'(z_k)} = \sum_{k=1}^{n} \sum_{j=1}^{n-1} \frac{1}{z_k - w_j} = \sum_{j=1}^{n-1} \sum_{k=1}^{n} \frac{(-1)}{w_j - z_k} = \sum_{j=1}^{n-1} \Big(- \frac{P'(w_j)}{P(w_j)} \Big) = 0.$$

□

The zeroes of $\alpha(z)$ are distinct because for every root z_0 of α' (which satisfies $\alpha(z_0)' = nz_0^{n-1} + 1 = 0$) we have

$$\alpha(z_0) = z_0^n + z_0 - 1 = \frac{n-1}{n} z_0 - 1 = \frac{n-1}{n^{1+1/(n-1)}} \epsilon_{n-1}^k - 1 \neq 0\,,$$

where ϵ_{n-1} is a root of -1 of degree $n-1$. Thus Lemma 2 can be applied and it gives

$$0 = \sum_{k=1}^{n} \frac{\alpha''(\lambda_k)}{\alpha'(\lambda_k)} = \sum_{k=1}^{n} \frac{n(n-1)\lambda_k^{n-2}}{n\lambda_k^{n-1} + 1} = n(n-1) \sum_{k=1}^{n} \frac{\lambda_k^{n-2}}{n\lambda_k^{n-1} + 1} = n(n-1) \operatorname{tr}\left(\left(nA^{n-1} + I\right)^{-1} A^{n-2} \right)\,,$$

which proves the desired equality. □