The 32$^{\text{nd}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1$^{\text{st}}$ May 2025
Category II

Problem 1 *Let* $x_0 = a$, $x_1 = b$, $x_2 = c$ *for given numbers* $a, b, c \in \mathbb{R}$, *and let* $x_{n+2} = \dfrac{x_n + x_{n-1}}{2}$ *for* $n \geq 1$. *Show that the sequence* $(x_n)_{n=0}^{\infty}$ *converges, and find its limit.* [10 points]

Problem 2 *Let* A, B *be two* $n \times n$ *complex matrices of the same rank, and let* $k \in \mathbb{N}$. *Prove that* $A^{k+1}B^k = A$ *if and only if* $B^{k+1}A^k = B$. [10 points]

Problem 3 *Evaluate the integral*

$$\int_0^{\infty} \frac{\log(x+2)}{x^2 + 3x + 2} \, \mathrm{d}x \,.$$

[10 points]

Problem 4 *Let* $D = \{z : |z| < 1\}$ *be the unit disk in the complex plane, and suppose that* $f \colon D \to D$ *is a holomorphic function fulfilling the property* $\lim\limits_{|z| \to 1} |f(z)| = 1$. *Let the Taylor series of* f *be* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. *Prove that* $\sum\limits_{n=0}^{\infty} n|a_n|^2$ *is equal to the number of zeros of* f *(counted with multiplicities).* [10 points]