The 32nd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1st May 2025
Category I

Problem 1 *Let $a \geq 2$ be an integer. Prove that there exists a positive integer b with the following property: For each positive integer n, there is a prime number $p = p(a,b,n)$ such that $a^n + b$ is divisible by p but not divisible by p^2.* [10 points]

Problem 2 *Find all real numbers $x > 1$ that satisfy the equality*

$$\left\lfloor \frac{n+1}{x} \right\rfloor = n - \left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{n}{x} \right\rfloor}{x} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{n}{x} \right\rfloor}{x} \right\rfloor}{x} \right\rfloor + \dots$$

for all positive integers n.

Here, $\lfloor t \rfloor$ is the integer part of t, i.e. the integer such that $\lfloor t \rfloor \leq t < \lfloor t \rfloor + 1$. [10 points]

Problem 3 *Let us call a sequence $(b_1, b_2, \dots)$ of positive integers* fast-growing *if $b_{n+1} \geq b_n + 2$ for all $n \geq 1$. Also, for a sequence $a = \big(a(1), a(2), \dots\big)$ of real numbers and a sequence $b = (b_1, b_2, \dots)$ of positive integers, let us denote*

$$S(a,b) = \sum_{n=1}^{\infty} \big| a(b_n) + a(b_n + 1) + \cdots + a(b_{n+1} - 1) \big| .$$

1. *Do there exist two fast-growing sequences $b = (b_1, b_2, \dots)$, $c = (c_1, c_2, \dots)$ such that for every sequence $a = \big(a(1), a(2), \dots\big)$, if all the series*

$$\sum_{n=1}^{\infty} a(n)\,, \quad S(a,b) \quad \text{and} \quad S(a,c)$$

are convergent, then the series $\sum_{n=1}^{\infty} |a(n)|$ is also convergent?

2. *Do there exist three fast-growing sequences $b = (b_1, b_2, \dots)$, $c = (c_1, c_2, \dots)$, $d = (d_1, d_2, \dots)$ such that for every sequence $a = \big(a(1), a(2), \dots\big)$, if all the series*

$$S(a,b)\,, \quad S(a,c) \quad \text{and} \quad S(a,d)$$

are convergent, then the series $\sum_{n=1}^{\infty} |a(n)|$ is also convergent?

[10 points]

Problem 4 *Let $n \geq 2$ be an integer, and let A be an $n \times n$ real matrix with minimal polynomial $x^n + x - 1$. Show that*

$$\operatorname{tr}\big((nA^{n-1} + I)^{-1} A^{n-2}\big) = 0\,.$$

Here, $\operatorname{tr}(B) = \sum_{i=1}^{n} b_{ii}$ denotes the trace of the matrix $B = (b_{ij})_{i,j=1}^{n}$.

[10 points]