The 31^{st} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13^{th} April 2024
Category I

Problem 1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Prove that*

$$\left| f(1) - \int_0^1 f(x)\,\mathrm{d}x \right| \le \frac{1}{2} \max_{x\in[0,1]} |f'(x)|\,.$$

[10 points]

Problem 2 *Let n be a positive integer and let A, B be two complex nonsingular $n \times n$ matrices such that*

$$A^2B - 2ABA + BA^2 = 0\,.$$

Prove that the matrix $AB^{-1}A^{-1}B - I_n$ is nilpotent. (Here I_n denotes the $n \times n$ identity matrix. A matrix X is called nilpotent if there exists a positive integer k such that $X^k = 0$.) [10 points]

Problem 3 *Let n be a positive integer and let G be a simple undirected graph on n vertices. Let d_i be the degree of its i-th vertex, $i = 1, \ldots, n$. Denote $\Delta = \max d_i$. Prove that if*

$$\sum_{i=1}^{n} d_i^2 > n\Delta(n - \Delta)$$

then G contains a triangle. (A graph is called simple if there are no loops and no multiple edges between any pair of vertices.) [10 points]

Problem 4 *Let $p > 2$ be a prime and let*

$$\mathcal{A} = \bigl\{ n \in \mathbb{N} : 2p \mid n \text{ and } p^2 \nmid n \text{ and } n \mid 3^n - 1 \bigr\}\,.$$

Prove that

$$\limsup_{k\to\infty} \frac{\bigl|\mathcal{A} \cap [1,k]\bigr|}{k} \le \frac{2\log 3}{p \log p}\,.$$

[10 points]