The 29th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2019
Category II

Problem 1

a) *Is it true that for every non-empty set A and every associative operation* $*\colon A \times A \to A$ *the conditions*

$$x * x * y = y \qquad \text{and} \qquad y * x * x = y \qquad \text{for every } x, y \in A$$

imply commutativity of $*$*?*

b) *Is it true that for every non-empty set A and every associative operation* $*\colon A \times A \to A$ *the condition*

$$x * x * y = y \qquad \text{for every } x, y \in A$$

implies commutativity of $*$*?*

[Paulius Drungilas and Artūras Dubickas / Vilnius University]

Solution

(i) Yes. Substituting $y \to y * x * y$ into $x * x * y = y$, we obtain

$$x * (x * y) * (x * y) = y * x * y.$$

In view of $y * x * x = y$ this implies $x = y * x * y$. "Multiplying" both sides of this equality by y (on the left) and using the first equality $x * x * y = y$ in the form $y * y * (x * y) = x * y$, we deduce that

$$y * x = y * y * x * y = x * y.$$

(ii) No. Indeed, consider the set $A = \{a, b\}$, where $a \neq b$, and the operation $*$, defined as follows: $a * a = a$, $b * b = b$, $a * b = b$ and $b * a = a$. One can easily verify that this operation is associative and that $x * x * y = y$ for all $x, y \in A = \{a, b\}$. However, $*$ is not commutative, since $a * b \neq b * a$.

□

Problem 2 *Find all twice differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that*

$$f''(x)\cos(f(x)) \geq (f'(x))^2 \sin(f(x)) \qquad \textit{for every } x \in \mathbb{R}\,. \tag{1}$$

[Orif Ibrogimov, Karim Rakhimov / Czech Technical University of Prague, University of Pisa]

Solution Obviously, every constant function satisfies (1). We show that there are no non-constant functions obeying (1). To this end, we observe that (1) is equivalent to the inequality $(\sin(f(x)))'' \geq 0$. Hence, $g(x) := \sin(f(x))$ is convex and thus

$$g(x) \geq g(y) + g'(y)(x-y) \qquad \forall x, y \in \mathbb{R}. \tag{2}$$

If f is non-constant and differentiable, then g cannot be a constant function, i.e. there is $y_0 \in \mathbb{R}$ such that $g'(y_0) \neq 0$. Setting $y = y_0$ in (2) and letting $x \to \operatorname{sgn} g'(y_0) \cdot \infty$, we conclude that g is an unbounded function. This is a contradiction. $\square$

The 29^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29^{th} March 2019
Category II

Problem 3 *Let p be an even non-negative continuous function with $\int_{\mathbb{R}} p(x)\,\mathrm{d}x = 1$ and let n be a positive integer. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent identically distributed random variables with density function p. Define*

$$\begin{aligned} X_0 &= 0\,, \\ X_1 &= X_0 + \xi_1\,, \\ X_2 &= X_1 + \xi_2\,, \\ &\vdots \\ X_n &= X_{n-1} + \xi_n\,. \end{aligned}$$

Prove that the probability that all the random variables $X_1, X_2, \ldots, X_{n-1}$ lie between X_0 and X_n equals $\frac{1}{n}$.

[Fedor Petrov / Saint-Petersburg State University]

Solution Denote by $I(\xi_1, \xi_2, \ldots, \xi_n)$ the indicator of the event "all the locations $X_1, X_2, \ldots, X_{n-1}$ lie between X_0 and X_n". Assuming that no linear combination of ξ_i's with coefficients $0, \pm 1$ equals to 0 (that happens almost surely) we have

$$I(\xi_1, \xi_2, \ldots, \xi_n) + I(\xi_2, \xi_3, \ldots, \xi_n, -\xi_1) + I(\xi_3, \xi_4, \ldots, \xi_n, -\xi_1, -\xi_2) + \cdots + I(\xi_n, -\xi_1, \ldots, -\xi_{n-1}) = 1.$$

Taking the expectation and using the symmetry of the distribution and independence we get $nI(\xi_1, \ldots, \xi_n) = 1$ as desired. For proving this identity we write down the numbers $\xi_1, \xi_2, \ldots, \xi_n, -\xi_1, -\xi_2, \ldots, -\xi_n$ around the circle (in this order). Note that $I(\xi_1, \xi_2, \ldots, \xi_n) = 1$ if and only if all $2n-1$ sums

$$\xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \cdots + \xi_n + (-\xi_1) + (-\xi_2) + \cdots + (-\xi_{n-1})$$

have the same sign. By G. M. Raney's lemma there exist only two out of $2n$ places on the circle with such a property. $\square$

The 29th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2019
Category II

Problem 4 *Let $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$. Let $n \geq 1$ and let $a_1, \ldots, a_n \in D$ be distinct complex numbers. Define*

$$f(z) = z \cdot \prod_{j=1}^{n} \frac{z - a_j}{z - \overline{a_j}}.$$

Prove that f' has at least one root in D. [Géza Kós / Loránd Eötvös University, Budapest]

Solution Let $b_1, \ldots, b_{2n}$ be the roots of f'. First we prove that

$$\sum \frac{1}{b_j} = 2 \sum \frac{1}{a_j}. \tag{1}$$

Let

$$f(z) = A\frac{g(z)}{h(z)} = A\frac{z + c_1 z^2 + c_2 z^3 + \ldots}{1 + \overline{c_1} z + \overline{c_2} z^2 + \ldots}.$$

where

$$c_1 = -\sum \frac{1}{a_j}. \tag{2}$$

Now consider

$$f'(z) = A\frac{g'(z)h(z) - g(z)h'(z)}{h^2(z)} = A\frac{1 + 2c_1 z + ...}{h^2(z)};$$

we can read that

$$2c_1 = -\sum \frac{1}{b_j}. \tag{3}$$

The results (2) and (3) together prove (1).

Since every a_j lies in D, we have

$$\sum \operatorname{Im} \frac{1}{b_j} = 2 \sum \operatorname{Im} \frac{1}{a_j} > 0,$$

so there is a b_j with $\operatorname{Im} b_j > 0$.

Hence it suffices that f' has no root in the closed upper-left quarter plane. That can be done by the following observation: For every z with $\operatorname{Re} z \leq 0$ and $\operatorname{Im} z \geq 0$,

$$\operatorname{Re} \frac{f'(z)}{f(z)} = \operatorname{Re} \left(\frac{1}{z} + \sum \left(\frac{1}{z - a_j} - \frac{1}{z - \overline{a_j}} \right) \right) = \frac{\operatorname{Re z}}{|z|^2} + \sum \big(\operatorname{Re}(z - a_j) \big) \left(\frac{1}{|z - a_j|^2} - \frac{1}{|z - \overline{a_j}|^2} \right) < 0.$$

□