The 23rd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 12th April 2013
Category II

Problem 1 *Let S_n denote the sum of the first n prime numbers. Prove that for any n there exists the square of an integer between S_n and S_{n+1}.*

Solution We have

$$\sqrt{x} < m < \sqrt{y} \Rightarrow x < m^2 < y,$$

so if $\sqrt{y} - \sqrt{x} > 1$, there is certainly a square between x and y.
We have

$$\sqrt{y} - \sqrt{x} > 1 \Rightarrow y - x > 1 + 2\sqrt{x},$$

hence it suffices to prove

$$S_{n+1} - S_n > 1 + 2\sqrt{S_n}.$$

For $n = 1, 2, 3, 4$ the assertion can be seen directly. For $n \geq 5$, we use

$$S_n < 1 + 3 + 5 + \ldots + p_n,$$

where the sum contains all odd integers up to p_n. Their sum equals $1/4(1+p_n)^2$, so it follows that $2\sqrt{S_n} < 1+p_n$. As p_{n+2} is at least $p_n + 2$, we get $S_{n+1} - S_n > 1 + 2\sqrt{S_n}$ as desired. □

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Problem 2 *An n-dimensional cube is given. Consider all the segments connecting any two different vertices of the cube. How many distinct intersection points do these segments have (excluding the vertices)?*

Solution We may think that every vertex of the cube has a view $(\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$. A cross-point of two segments has a view $(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \{0, \frac{1}{2}, 1\}$. For example, if $A = (0, 0, 0, 1, 1)$, $B = (1, 0, 0, 0, 1)$, $C = (1, 0, 0, 1, 1)$, $D = (0, 0, 0, 0, 1)$ then $AB \cap CD = (\frac{1}{2}, 0, 0, \frac{1}{2}, 1)$. However a row containing less than 2 of $\frac{1}{2}$ may be not a cross-point. Therefore, there are exactly $3^n - 2^n - n2^{n-1}$ of cross-points. $\square$

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Problem 3 *Prove that there is no polynomial P with integer coefficients such that $P(\sqrt[3]{5}+\sqrt[3]{25}) = 5+\sqrt[3]{5}$.*

Solution First we prove two lemmas.

Lemma 1. *There is no polynomial $w(x) = ax + b$ with integer coefficients such that $w(\sqrt[3]{5}+\sqrt[3]{25}) = 5+\sqrt[3]{5}$;*

Proof Assume on the contrary that such a polynomial $w(x) = ax + b$ exists. Since $\sqrt[3]{5}$ and $\sqrt[3]{25}$ are irrational, it follows that $a \neq 0$ and $a \neq 1$. Furthermore, one has

$$\begin{aligned}
&a(\sqrt[3]{5}+\sqrt[3]{25}) + b = 5+\sqrt[3]{5} \Longrightarrow (a-1)\sqrt[3]{5} + a\sqrt[3]{25} \in \mathbb{Q} \\
&\Longrightarrow \Big((a-1)\sqrt[3]{5} + a\sqrt[3]{25}\Big)^2 \in \mathbb{Q} \Longrightarrow (a-1)^2\sqrt[3]{25} + 5a^2\sqrt[3]{5} \in \mathbb{Q} \\
&\Longrightarrow \frac{5a^2}{(1-a)}\Big((a-1)\sqrt[3]{5} + a\sqrt[3]{25}\Big) + \Big((a-1)^2\sqrt[3]{25} + 5a^2\sqrt[3]{5}\Big) \in \mathbb{Q} \\
&\Longrightarrow \left(\frac{(a-1)^3 - 5a^3}{(a-1)}\right)\sqrt[3]{25} \in \mathbb{Q} \Longrightarrow \sqrt[3]{25} \in \mathbb{Q},
\end{aligned}$$

which contradicts the fact that $\sqrt[3]{25} \in n\mathbb{Q}$, where $\mathbb{Q}$ and $n\mathbb{Q}$ denote the set of rational and irrational numbers, respectively. This completes the proof of the lemma. □

Lemma 2. *There exists exactly one polynomial $w(x)$ of degree two and rational coefficients such that $w(\sqrt[3]{5}+\sqrt[3]{25}) = 5+\sqrt[3]{5}$;*

Proof Consider a polynomial $w(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{Q}$. Then

$$\begin{aligned}
&w(\sqrt[3]{5}+\sqrt[3]{25}) = 5+\sqrt[3]{5} \Longleftrightarrow a(\sqrt[3]{5}+\sqrt[3]{25})^2 + b(\sqrt[3]{5}+\sqrt[3]{25}) + c = 5+\sqrt[3]{5} \\
&\Longleftrightarrow \begin{cases} a+b &= 0 \\ 5a+b &= 1 \\ 10a+c &= 5 \end{cases} \Longleftrightarrow \begin{cases} a &= 1/4 \\ b &= -1/4 \\ c &= 10/4 \end{cases}
\end{aligned}$$

This implies that there exists only one polynomial $w(x)$ with the required properties, i.e.,

$$w(x) = \frac{1}{4}x^2 - \frac{1}{4}x + \frac{10}{4} \text{ and } w(\sqrt[3]{5}+\sqrt[3]{25}) = 5+\sqrt[3]{5},$$

which completes the proof of the second lemma.

□

Now we are ready to solve the problem. Let $x_0 := \sqrt[3]{5}+\sqrt[3]{25}$. Then

$$x_0^3 = (\sqrt[3]{5}+\sqrt[3]{25})^3 = 5 + 3\sqrt[3]{5^4} + 3\sqrt[3]{5^5} + 25 = 30 + 15\sqrt[3]{5} + 15\sqrt[3]{5} = 15x_0 + 30.$$

We put $Q(x) := x^3 - 15x - 30$. Then $Q(x_0) = 0$. Assume on the contrary that such a polynomial $P(x)$ exists. Then there exist two polynomials $R(x)$ and $w(x)$ with integer coefficients such that

$$P(x) = Q(x)R(x) + w(x),$$

where the degree $\deg w(x)$ of $w(x)$ is less than or equal 2. Consequently we obtain

$$5+\sqrt[3]{5} = P(\sqrt[3]{5}+\sqrt[3]{25}) = Q(\sqrt[3]{5}+\sqrt[3]{25})R(\sqrt[3]{5}+\sqrt[3]{25}) + w(\sqrt[3]{5}+\sqrt[3]{25}) = w(\sqrt[3]{5}+\sqrt[3]{25}).$$

From this it follows that there exists a polynomial $w(x)$ of degree less than or equal 2 with integer coefficients such that

$$w(\sqrt[3]{5}+\sqrt[3]{25}) = 5+\sqrt[3]{5},$$

a contradiction with Lemma 1 and Lemma 2. This completes the solution. □

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Problem 4 *Let $\mathcal{F}$ be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with the property*

$$\left| \int_0^x \frac{f(t)}{\sqrt{x-t}} \,\mathrm{d}t \right| \le 1 \quad \text{for all } x \in (0,1]\,.$$

Compute $\sup\limits_{f\in\mathcal{F}} \left| \int_0^1 f(x)\,\mathrm{d}x \right|$.

Solution We will use the following lemma.

Lemma *For every functions $f \in L_1[0,1]$,*

$$\int_0^1 \left(\int_0^x \frac{f(t)\mathrm{d}t}{\sqrt{x-t}} \right) \frac{\mathrm{d}x}{\sqrt{1-x}} = \pi \int_0^1 f.$$

Proof Changing the order of integration then substituting $t = -1 + 2\frac{x-t}{1-t}$,

$$\begin{aligned} \int_0^1 \left(\int_0^x \frac{f(t)\mathrm{d}t}{\sqrt{x-t}} \right) \frac{\mathrm{d}x}{\sqrt{1-x}} &= \int_0^1 f(t) \left(\int_t^1 \frac{\mathrm{d}x}{\sqrt{(x-t)(1-x)}} \right) \mathrm{d}t \\ &= \int_0^1 f(t) \left(\int_{-1}^1 \frac{\mathrm{d}t}{\sqrt{(1+t)(1-t)}} \right) \mathrm{d}t = \pi \int_0^1 f. \end{aligned}$$

□

Now, by Lemma, for all $f \in \mathcal{F} \subset L_1[0,1]$ we have

$$\left| \int_0^1 f \right| \le \frac{1}{\pi} \int_0^1 \left| \int_0^x \frac{f(t)\mathrm{d}t}{\sqrt{x-t}} \right| \frac{\mathrm{d}x}{\sqrt{1-x}} \le \frac{1}{\pi} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} = \frac{2}{\pi}$$

so $\sup_{f\in\mathbb{F}} \left| \int_0^1 f \right| \le \frac{2}{\pi}$.

For the function $g(x) = \dfrac{1}{\pi\sqrt{x}}$ we have

$$\int_0^x \frac{g(t)\mathrm{d}t}{\sqrt{x-t}} = \frac{1}{\pi} \int_0^x \frac{\mathrm{d}t}{\sqrt{t(x-t)}} = 1.$$

Define a sequence $f_1, f_2, \dots$ of $[0,1] \to \mathbb{R}$ functions as $f_n(x) = \dfrac{1}{\pi\sqrt{x+\frac{1}{n}}}$. Then $f_n \in C[0,1]$ and $0 < f \le g$, so $f_n \in \mathcal{F}$. As $f_n(x) \to g(x)$ pointwise, we have $\int_0^1 f_n \to \int_0^1 g = \frac{2}{\pi}$.

Hence, $\sup_{f\in\mathcal{F}} \left| \int_0^1 f \right| = \frac{2}{\pi}$. □