The 19$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1$^{\text{st}}$ April 2009
Category I

Problem 1 *Let ABC be a non-degenerate triangle in the euclidean plane. Define a sequence $(C_n)_{n=0}^{\infty}$ of points as follows: $C_0 := C$, and C_{n+1} is the center of the incircle of the triangle ABC_n. Find $\lim_{n\to\infty} C_n$.*

[10 points]

Problem 2 *Prove that the number*

$$2^{2^k - 1} - 2^k - 1$$

is composite (not prime) for all positive integers $k > 2$. [10 points]

Problem 3 *Let k and n be positive integers such that $k \le n-1$. Let $S := \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of S. Prove that it is possible to color some elements of S using two colors, red and blue, such that the following conditions are satisfied:*

(i) Each element of S is either left uncolored or is colored red or blue.

(ii) At least one element of S is colored.

(iii) Each set A_i ($i = 1, 2, \ldots, k$) is either completely uncolored or it contains at least one red and at least one blue element.

[10 points]

Problem 4 *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence $(a_n)_{n=1}^{\infty}$* covers *the set of positive integers if for any positive integer m there exists a positive integer k such that $\sum_{n=1}^{\infty} a_n^k = m$.*

a) Does there exist a sequence of real positive numbers which covers the set of positive integers?

b) Does there exist a sequence of real numbers which covers the set of positive integers?

[10 points]

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