Problem j13-I-1/j13-I-15. *Let $d(k)$ be the number of all natural divisors of a number $k \in \mathbb{N}$. Prove that for any $n_0 \in \mathbb{N}$ the sequence $\big(d(n^2+1)\big)_{n=n_0}^{\infty}$ is not strictly monotone.*

(Vilnius University)

Solution. Note that $d(n^2+1) < n$ for all even n. Indeed, the number n^2+1 is not square and so it is possible to split the set of all its divisors into pairs $\{\, d,\ (n^2+1)/d \,\}$ where $d < n$ and d is odd. The number of divisors in all such pairs does not exceed n.

Let us assume that starting from some $n_0 \in \mathbb{N}$, the sequence is strictly monotone. For $d(n^2+1)$ is always even, we get

$$d\big((n+1)^2+1\big) \geq d\big(n^2+1\big) + 2$$

or, in general,

$$d\big((n+k)^2+1\big) \geq d\big(n^2+1\big) + 2k$$

for any natural numbers $n \geq n_0$ and $k \geq 1$. Let $N \geq n_0$ (e.g., $N = n_0$). Taking any $s \geq N - d(N^2+1)$ (such that $N+s$ is even), we get

$$d\big((N+s)^2+1\big) \geq d\big(N^2+1\big) + 2s \geq N+s,$$

which is a contradiction with $d\big((N+s)^2+1\big) < N+s$. $\square$

Problem j13-I-2/j13-I-19. *Let $A = [a_{i,j}]$ be an $m \times n$ real matrix with at least one non-zero element. For each $i \in \{1,\dots,m\}$ let $R_i := \sum_{j=1}^{n} a_{i,j}$ (the sum of the i-th row of A) and for each $j \in \{1,\dots,n\}$ let $C_j := \sum_{i=1}^{m} a_{i,j}$ (the sum of the j-th column of A). Prove that there exist indices $k \in \{1,\dots,m\}$ and $l \in \{1,\dots,n\}$ such that*

$$a_{k,l} > 0\,, \qquad R_k \geq 0\,, \qquad C_l \geq 0\,,$$

or

$$a_{k,l} < 0\,, \qquad R_k \leq 0\,, \qquad C_l \leq 0\,.$$

(University of Zagreb)

Solution. Consider the following sets of indices (some of them may be empty):

$$\begin{aligned} I^+ &:= \left\{\, i \in \{1,\dots,m\} \mid R_i \geq 0 \,\right\}, \\ I^- &:= \left\{\, i \in \{1,\dots,m\} \mid R_i < 0 \,\right\}, \\ J^+ &:= \left\{\, j \in \{1,\dots,n\} \;\mid C_j > 0 \,\right\}, \\ J^- &:= \left\{\, j \in \{1,\dots,n\} \;\mid C_j \leq 0 \,\right\}. \end{aligned}$$

Suppose that the statement of the problem does not hold. Then (but not equivalently) we have $a_{i,j} \leq 0$ for every $(i,j) \in I^+ \times J^+$ and we have $a_{i,j} \geq 0$ for every $(i,j) \in I^- \times J^-$. Let us write the sum $\sum_{(i,j)\in I^- \times J^+} a_{i,j}$ in two different ways:

$$\sum_{(i,j)\in I^-\times J^+} a_{i,j} = \sum_{i\in I^-}\left(\sum_{j=1}^{n} a_{i,j} - \sum_{j\in J^-} a_{i,j}\right) = \sum_{i\in I^-} R_i - \sum_{(i,j)\in I^-\times J^-} a_{i,j} \leq 0\,,$$

$$\sum_{(i,j)\in I^-\times J^+} a_{i,j} = \sum_{j\in J^+}\left(\sum_{i=1}^{m} a_{i,j} - \sum_{i\in I^+} a_{i,j}\right) = \sum_{j\in J^+} C_j - \sum_{(i,j)\in I^+\times J^+} a_{i,j} \geq 0\,.$$

Therefore, $\sum_{(i,j)\in I^-\times J^+} a_{i,j} = 0$ and we have only equalities in the two formulae above. This is only possible if $\sum_{i\in I^-} R_i = 0$ and $\sum_{j\in J^+} C_j = 0$, so $I^- = \emptyset$ and $J^+ = \emptyset$,† which means $R_i \geq 0$ for all $i = 1, \dots, m$ and $C_j \leq 0$ for all $j = 1, \dots, n$. Moreover, from

$$0 \leq \sum_{i=1}^{m} R_i = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i,j} = \sum_{j=1}^{n} C_j \leq 0,$$

we conclude $R_i = 0$ for $i = 1, \dots, m$ and $C_j = 0$ for $j = 1, \dots, n$. Since A is a non-zero matrix, there are indices k and l such that $a_{k,l} \neq 0$, but $R_k = 0$ and $C_l = 0$, which leads to a contradiction with the assumption that the statement of the problem is false. □

† If $I^- \neq \emptyset$, then $\sum_{(i,j)\in I^-\times J^+} a_{i,j} \leq \sum_{i\in I^-} R_i < 0$ — a contradiction. We can argue similarly to show $J^+ = \emptyset$.

Problem j13-I-3/j13-I-9. *Find the limit*

$$\lim_{n\to\infty}\sqrt{1+2\sqrt{1+3\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\ .$$

(Dr. Moubinool Omarjee, Paris†)

Solution. Let

$$u_{m,n}=\sqrt{1+m\sqrt{1+(m+1)\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\ .$$

We have

$$u_{m,n}^2 = 1+mu_{m+1,n}\ ,$$
$$u_{m,n}^2-(m+1)^2 = m\big(u_{m+1,n}-(m+2)\big).$$

Using the equality $|a-b|=|a^2-b^2|/|a+b|$ and inequality $u_{m,n}+m+1\geq m+2$, we get

$$\big|u_{m,n}-m-1\big|\leq\frac{m}{m+2}\big|u_{m+1,n}-(m+2)\big|.$$

We deduce that

$$|u_{2,n}-3|\leq\frac{2}{4}\cdot\frac{3}{5}\cdot\dots\cdot\frac{n-1}{n+1}\cdot|u_{n-1,n}-n|,$$
$$|u_{2,n}-3|\leq\frac{6}{n(n+1)}\left(\sqrt{1+(n-1)\sqrt{1+n}}-n\right)=O\left(\frac{1}{n}\right).$$

So we get

$$\lim_{n\to\infty}u_{2,n}=3.$$

□

† This problem is formally proposed by the University of Ostrava.

Problem j13-I-4/j13-I-12. *Let A and B be complex hermitian 2×2 matrices with pairs of eigenvalues (α_1, α_2) and (β_1, β_2), respectively. Determine all possible pairs (γ_1, γ_2) of eigenvalues of the matrix $C = A + B$. (A matrix $A = [a_{i,j}]$ is hermitian if and only if $a_{i,j} = \overline{a_{j,i}}$ for all i, j.)* (Charles University in Prague)

Solution. Recall that all eigenvalues of a hermitian matrix are real numbers and that there exists an orthonormal basis consisting of eigenvectors of the matrix. As we can add a suffitiently large multiple of the identity matrix to both matrices A and B, we can suppose wlog that $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and also $\gamma_1, \gamma_2 > 0$.

Let us also wlog suppose $\alpha_1 \geq \alpha_2,\ \ \beta_1 \geq \beta_2,\ \ \gamma_1 \geq \gamma_2$ and $\alpha_1 - \alpha_2 \geq \beta_1 - \beta_2$. By easy arguments, we can see

$$\gamma_1 + \gamma_2 = \operatorname{Tr} C = \operatorname{Tr} A + \operatorname{Tr} B = \alpha_1 + \alpha_2 + \beta_1 + \beta_2.$$

Further, it holds that

$$\gamma_1 \leq \alpha_1 + \beta_1, \qquad \gamma_2 \geq \alpha_2 + \beta_2.$$

(The first inequality can be seen if we rewrite it slightly: $\gamma_1 = \|C\| \leq \|A\| + \|B\| = \alpha_1 + \beta_1$. The second inequality follows if we consider the equality above and the first inequality together. — Alternatively, $\gamma_1 = \max(Cx, x)/(x, x) \leq \max(Ax, x)/(x, x) + \max(Bx, x)/(x, x) = \alpha_1 + \beta_1$ and $\gamma_2 = \min(Cx, x)/(x, x) \geq \min(Ax, x)/(x, x) + \min(Bx, x)/(x, x) = \alpha_2 + \beta_2$.) Later we will also prove the inequalities

$$\gamma_1 \geq \alpha_1 + \beta_2, \qquad \gamma_2 \leq \beta_1 + \alpha_2$$

(in fact, it suffices to prove only the first one because the second one follows if we use the equality given above).

From these inequalities, we can see that $\gamma_1 \in [\alpha_1 + \beta_2, \alpha_1 + \beta_1]$. (The value of γ_2 has to be "complementary" to obtain the right value of the sum $\gamma_1 + \gamma_2$. It also worths noting that even if $\gamma_1 = \alpha_1 + \beta_2$, then still $\gamma_1 \geq \gamma_2 = \beta_1 + \alpha_2$. This follows from the assumption $\alpha_1 - \alpha_2 \geq \beta_1 - \beta_2$.) We will show that γ_1 can assume any value from the given interval $[\alpha_1 + \beta_2, \alpha_1 + \beta_1]$. Consequently, the set of all possible pairs (γ_1, γ_2) of eigenvalues of the matrix $C = A + B$ is

$$\big\{ (\gamma_1, \gamma_2) : \alpha_1 + \beta_2 \leq \gamma_1 \leq \alpha_1 + \beta_1,\ \gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \big\}.$$

To see this, let us put

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \qquad P(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The matrix A obviously has eigenvalues (α_1, α_2). The matrix $B(t) = P^{-1}(t) B P(t)$ obviously has eigenvalues (β_1, β_2). If we note that $P^{-1}(t) = P^T(t)$ and define the matrix $C(t) = A + B(t)$, we have

$$C(0) = A + B = \begin{pmatrix} \alpha_1 + \beta_1 & 0 \\ 0 & \alpha_2 + \beta_2 \end{pmatrix}, \qquad C(\tfrac{\pi}{2}) = \begin{pmatrix} \alpha_1 + \beta_2 & 0 \\ 0 & \alpha_2 + \beta_1 \end{pmatrix}.$$

The matrix $C(0)$ has the eigenvalue $\gamma_1(0) = \alpha_1 + \beta_1$. (Note that $\gamma_1(0) \geq \gamma_2(0) = \alpha_2 + \beta_2$.) The matrix $C(\pi/2)$ has the eigenvalue $\gamma_1(\pi/2) = \alpha_1 + \beta_2$. (Note that $\gamma_1(\pi/2) \geq \gamma_2(\pi/2) = \alpha_2 + \beta_1$.) As both eigenvalues (γ_1, γ_2) of a matrix C depend continuously on the coefficients of the matrix, we deduce that $\gamma_1(t)$ is a continuous function. Consequently, it assumes every value from the interval $[\alpha_1 + \beta_2, \alpha_1 + \beta_1]$, which we wanted to demonstrate.

Now it only remains to prove the inequality $\gamma_1 \geq \alpha_1 + \beta_2$ for any two complex hermitian matrices A and B. Let us recall that we still wlog suppose $\alpha_1 \geq \alpha_2 > 0,\ \ \beta_1 \geq \beta_2 > 0$ and $\gamma_1 \geq \gamma_2 > 0$. Let v_1 and v_2 denote the eigenvectors of the matrix A corresponding to the eigenvalues α_1 and α_2, respectively, and let w_1 and w_2 denote the eigenvectors of B corresponding to the eigenvalues β_1 and β_2, respectively. We can suppose that the bases

$\{v_1, v_2\}$ and $\{w_1, w_2\}$ are orthonormal. So there exists some unitary matrix $U = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$ such that

$$\begin{aligned} v_1 &= u_{11}w_1 + u_{12}w_2, \\ v_2 &= u_{21}w_1 + u_{22}w_2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} w_1 &= \overline{u_{11}}v_1 + \overline{u_{21}}v_2, \\ w_2 &= \overline{u_{12}}v_1 + \overline{u_{22}}v_2. \end{aligned}$$

We will estimate γ_1 in the following way. First,

$$\gamma_1 = \sup\big\{ \|Cx\| : \|x\| = 1 \big\} \geq \|Cv_1\|$$

where $\|\cdot\|$ denotes the Euclidean norm. (Let us justify the formula. Recall that $\gamma_1 = \max_{\|x\|=1}(Cx, x)$. Obviously, γ_1^2 is the greater eigenvalue of C^2. Consequently, it follows that $\gamma_1^2 = \max_{\|x\|=1}(C^2x, x)$. As C is hermitian, we have $(C^2x, x) = x^*CCx = x^*C^*Cx = (Cx, Cx) = \|Cx\|^2$.) Second,

$$\begin{aligned} Cv_1 = (A+B)v_1 = \alpha_1 v_1 + \beta_1 u_{11} w_1 + \beta_2 u_{12} w_2 = (\alpha_1 + \beta_2)v_1 + (\beta_1 - \beta_2)u_{11}w_1 = \\ = \big(\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u_{11}}\big)v_1 + (\beta_1 - \beta_2)u_{11}\overline{u_{21}}v_2. \end{aligned}$$

As the vectors v_1 and v_2 are orthonormal and $(\beta_1 - \beta_2)u_{11}\overline{u_{11}} \geq 0$, we conclude

$$\begin{aligned} \gamma_1 \geq \|Cv_1\| = \sqrt{\big|\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u_{11}}\big|^2 + \big|(\beta_1 - \beta_2)u_{11}\overline{u_{21}}\big|^2} \geq \\ \geq \sqrt{\big|\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u_{11}}\big|^2} \geq \alpha_1 + \beta_2. \end{aligned}$$

□

Problem j13-II-1/j13-II-51. *Two real square matrices A and B satisfy the conditions $A^{2002} = B^{2003} = I$ and $AB = BA$. Prove that $A + B + I$ is invertible. (The symbol I denotes the identity matrix.)* (University of Belgrade)

Solution. Let $(A + B + I)x = 0$ for some vector x, i.e., $(B + I)x = -Ax$. Then we have $-A^2x = A(B + I)x = (B + I)Ax = -(B + I)^2x$, and, continuing in this way, $(B + I)^kx = (-1)^kA^kx$. As $A^{2002} = I$, we get $(B + I)^{2002}x = x$, i.e.,

$$\big((B + I)^{2002} - I\big)x = \big(B^{2003} - I\big)x = 0.$$

(Recall $B^{2003} = I$.) In other words, taking that $p(t) = (t + 1)^{2002} - 1$ and $q(t) = t^{2003} - 1$ are polynomials, we have just got

$$p(B)x = q(B)x = 0.$$

But, since 2003 is a prime, $q(t)/(t - 1)$ is a primitive polynomial for all its roots, and therefore none of them is a root of the another monic polynomial $p(t)$ of degree 2002; further, the remained root $t = 1$ of $q(t)$ is not a root of $p(t)$, which implies that $p(t)$ and $q(t)$ are coprime.†

Since there exist non-zero polynomials $r(t)$ and $s(t)$ such that $r(t)p(t) - s(t)q(t) = 1$ (recall the Euclidean algorithm), we can conclude that $x = r(B)p(B)x - s(B)q(B)x = 0$, and so $A + B + I$ must be invertible indeed. □

† The polynomials $p(t)$ and $q(t)$ are really coprime (i.e. relatively prime). Here is another argument: Every polynomial (of degree ≥ 1) can be written as a product of factors of degree 1. In particular, $p(t) = (t + 1)^{2002} - 1 = \prod_{k=1}^{2002}(t - z_{p,k})$ and $q(t) = t^{2003} - 1 = \prod_{k=1}^{2003}(t - z_{q,k})$, where $z_{p,1}, \ldots, z_{p,2002}$ and $z_{q,1}, \ldots, z_{q,2003}$ are the roots of the polynomial p and q, respectively. Obviously, the polynomials p and q are relatively prime iff they have no root in common.

It is easy to see that the roots of q lie on the unit circle in the complex plane. Similarly, it is easy to see that all roots of p are on the circle with radius 1 and its centre at the point -1.

Thus, the intersections of the two circles,

$$\tfrac{\sqrt{2}}{2} \pm \mathrm{i}\tfrac{\sqrt{2}}{2} = \cos \pm\tfrac{3\pi}{2} + \mathrm{i}\sin \pm\tfrac{3\pi}{2} = \big(-1\big) + \big(\cos \pm\tfrac{\pi}{2} + \mathrm{i}\sin \pm\tfrac{\pi}{2}\big),$$

are the only possible common roots of q and p. But none of these two points is a root of q. It follows that p and q are coprime.

Problem j13-II-2/j13-I-17. *Let $\{D_1, D_2, \ldots, D_n\}$ be a set of disks (a disk is a circle with its interior) in the Euclidean plane and $a_{ij} = S(D_i \cap D_j)$ be the area of $D_i \cap D_j$. Prove that for any numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ the following inequality holds:*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \geq 0.$$

(Warsaw University)

Solution. Let $\chi_{D_i} : \mathbb{R}^2 \to \{0, 1\}$ be the characteristic function of the set D_i:

$$\chi_{D_i}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in D_i, \\ 0, & \text{if } (x, y) \notin D_i. \end{cases}$$

We have:

$$\chi_{D_i \cap Dj} = \chi_{D_i} \chi_{D_j},$$

$$S(D_i) = \int_{\mathbb{R}^2} \chi_{D_i}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^2} \chi_{D_i}^2(x, y) \, \mathrm{d}x \, \mathrm{d}y,$$

$$S(D_i \cap D_j) = \int_{\mathbb{R}^2} \chi_{D_i \cap D_j}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^2} \chi_{D_i}(x, y) \chi_{D_j}(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \int_{\mathbb{R}^2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \chi_{D_i}(x, y) x_j \chi_{D_j}(x, y) \, \mathrm{d}x \, \mathrm{d}y =$$

$$= \int_{\mathbb{R}^2} \big(x_1 \chi_{D_1}(x, y) + \cdots + x_n \chi_{D_n}(x, y) \big)^2 \, \mathrm{d}x \, \mathrm{d}y \geq 0.$$

□

Problem j13-II-3/j13-II-70. *A sequence $(a_n)_{n=0}^{\infty}$ of real numbers is defined recursively by*

$$a_0 := 0, \qquad a_1 := 1, \qquad a_{n+2} := a_{n+1} + \frac{a_n}{2^n}\,,\ n \ge 0.$$

Prove that

$$\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \cdot \prod_{k=1}^{n}(2^k - 1)}\,.$$

(University of Zagreb)

Remark. In fact, we will prove the following:

(a) The sequence $(a_n)_{n=0}^{\infty}$ is convergent.
(b) $\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} 1/\big(2^{n(n-1)/2} \cdot \prod_{k=1}^{n}(2^k-1)\big)$.
(c) The limit $\lim_{n\to\infty} a_n$ is an irrational number.

Solution. (a) Obviously, $a_n \ge 0$ for every $n \ge 0$. The sequence $(a_n)_{n=0}^{\infty}$ is increasing since $a_{n+2} - a_{n+1} = a_n/2^n \ge 0$ for every $n \ge 0$. It suffices to show that $(a_n)_{n=0}^{\infty}$ is bounded from above. For each $n \ge 0$, we have $a_{n+2} \le a_{n+1} + a_{n+1}/2^n = a_{n+1}(1 + 1/2^n)$. Using the inequality between geometric and arithmetic mean, for every $n \ge 1$ we obtain

$$a_{n+2} \le \prod_{k=0}^{n}\left(1 + \frac{1}{2^k}\right) = 2\prod_{k=1}^{n}\left(1 + \frac{1}{2^k}\right) \le 2\left(\frac{1}{n}\left(n + \sum_{k=1}^{n}\frac{1}{2^k}\right)\right)^n \le 2\left(\frac{n+1}{n}\right)^n \le 2\mathrm{e}.$$

(b) Consider the power series $\sum_{n=0}^{\infty} a_n z^n$. Since $\limsup_{n\to\infty}\sqrt[n]{|a_n|} \le \lim_{n\to\infty}\sqrt[n]{2\mathrm{e}} = 1$, its radius of convergence is $R \ge 1$. Therefore, on the open unit disc, with center at the origin, it converges to a holomorphic function $f(z) := \sum_{n=0}^{\infty} a_n z^n$. Inductively, we obtain $a_{n+2} = 1 + \sum_{k=0}^{n} a_k/2^k$ for any $n \ge 0$. So $\lim_{n\to\infty} a_n = 1 + \sum_{k=0}^{\infty} a_k/2^k = 1 + f\big(\frac{1}{2}\big)$ and we have to find $f\big(\frac{1}{2}\big)$.

Now we use the recurrent relation for $(a_n)_{n=0}^{\infty}$ to obtain a functional equation for f. We multiply $a_{n+2} := a_{n+1} + a_n/2^n$ by z^{n+2} and sum over all $n \ge 0$ to get

$$\sum_{n=0}^{\infty} a_{n+2} z^{n+2} = z\sum_{n=0}^{\infty} a_{n+1} z^{n+1} + z^2 \sum_{n=0}^{\infty} a_n \big(\tfrac{z}{2}\big)^n,$$

that is

$$f(z) - z = zf(z) + z^2 f\big(\tfrac{z}{2}\big),$$

or

$$(1-z)f(z) = z^2 f\big(\tfrac{z}{2}\big) + z \qquad \text{for } |z| < 1. \tag{1}$$

We substitute $z = 1/2^n$ for $n = 1, \ldots, N$ (where $N \ge 1$ is a fixed number) into (1), then multiply the n-th equality by some constant $s_n > 0$ and finally sum up those N equalities:

$$\begin{aligned}
\big(1-\tfrac{1}{2}\big)f\big(\tfrac{1}{2}\big) &= \big(\tfrac{1}{2}\big)^2 f\big(\tfrac{1}{4}\big) + \tfrac{1}{2}\,, && |\cdot s_1\,,\\
\big(1-\tfrac{1}{4}\big)f\big(\tfrac{1}{4}\big) &= \big(\tfrac{1}{4}\big)^2 f\big(\tfrac{1}{8}\big) + \tfrac{1}{4}\,, && |\cdot s_2\,,\\
&\vdots\\
\big(1-\tfrac{1}{2^n}\big)f\big(\tfrac{1}{2^n}\big) &= \big(\tfrac{1}{2^n}\big)^2 f\big(\tfrac{1}{2^{n+1}}\big) + \tfrac{1}{2^n}\,, && |\cdot s_n\,,\\
\big(1-\tfrac{1}{2^{n+1}}\big)f\big(\tfrac{1}{2^{n+1}}\big) &= \big(\tfrac{1}{2^{n+1}}\big)^2 f\big(\tfrac{1}{2^{n+2}}\big) + \tfrac{1}{2^{n+1}}\,, && |\cdot s_{n+1}\,,\\
&\vdots\\
\big(1-\tfrac{1}{2^N}\big)f\big(\tfrac{1}{2^N}\big) &= \big(\tfrac{1}{2^N}\big)^2 f\big(\tfrac{1}{2^{N+1}}\big) + \tfrac{1}{2^N}\,, && |\cdot s_N\,,\\
\hline
\frac{s_1}{2} f\big(\tfrac{1}{2}\big) &= \frac{s_N}{2^{2N}} f\big(\tfrac{1}{2^{N+1}}\big) + \sum\nolimits_{n=1}^{N} \frac{s_n}{2^n}\,.
\end{aligned}$$

To obtain the given result (namely, to achieve cancelling of the terms with $f\big(\frac{1}{2^n}\big)$ for $n = 2, \ldots, N$), we had to choose the numbers s_n so that

$$\big(1 - \tfrac{1}{2^{n+1}}\big)s_{n+1} = \big(\tfrac{1}{2^n}\big)^2 s_n, \qquad \text{for } n \ge 0. \tag{2a}$$

Let us put

$$s_0 := 1. \tag{2b}$$

It follows that $s_1 = 2$. Equalities (2b) and (2a) lead to

$$s_n = \prod_{k=0}^{n-1} \frac{s_{k+1}}{s_k} = \prod_{k=0}^{n-1} \frac{\left(\frac{1}{2^k}\right)^2}{1 - \frac{1}{2^{k+1}}} = \prod_{k=0}^{n-1} \frac{1}{2^{k-1}(2^{k+1}-1)} = \frac{1}{2^{\frac{n(n-1)}{2}-n} \prod_{k=1}^{n}(2^k-1)}$$

for every $n \geq 1$. Finally, we have

$$f\left(\tfrac{1}{2}\right) = \frac{s_N}{2^{2N}} f\left(\tfrac{1}{2^{N+1}}\right) + \sum_{n=1}^{N} \frac{s_n}{2^n} = \frac{f\left(\frac{1}{2^{N+1}}\right)}{2^{\frac{N(N-1)}{2}+N} \prod_{k=1}^{N}(2^k-1)} + \sum_{n=1}^{N} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)}.$$

The first term tends to 0 when $N \to \infty$, so

$$f\left(\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)}. \tag{3}$$

(c) The proof of $\lim_{n\to\infty} a_n \in \mathbb{R}\setminus\mathbb{Q}$ is based on the fact that the series in (3) converges "very rapidly". Suppose that its sum equals $\frac{p}{q}$ for some positive integers p and q. For each integer $N \geq 1$, denote

$$q_N := 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N}(2^k-1), \qquad p_N := q_N \sum_{n=1}^{N} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)}.$$

Obviously, p_N and q_N are positive integers. We manage to estimate $pq_N - qp_N$. We have

$$q_N = 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N}(2^k-1) < 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N} 2^k = 2^{N^2}$$

and

$$\frac{p}{q} - \frac{p_N}{q_N} = \sum_{n=N+1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} 2^{k-1}} =$$

$$= \sum_{n=N+1}^{\infty} \frac{1}{2^{n(n-1)}} \leq \sum_{m=N(N+1)}^{\infty} \frac{1}{2^m} = \frac{1}{2^{N^2+N-1}} < \frac{1}{2^{N-1} q_N}.$$

Thus, $0 < pq_N - qp_N < \frac{q}{2^{N-1}}$,† so $(pq_N - qp_N)_{N\geq 1}$ is a sequence of positive integers that converges to 0. This is a contradiction and we are done. $\square$

† It is easy to see from the definition of the numbers p_N that the sequence $\left(\frac{p_N}{q_N}\right)$ is strictly increasing to the limit $\frac{p}{q}$. Hence $\frac{p_N}{q_N} < \frac{p}{q}$, $qp_n < pq_N$, and $0 < pq_N - qp_N$. As the difference is integer, we have even $1 \leq pq_N - qp_N$.

Problem j13-II-4/j13-I-18. *Let $f, g\colon [0,1] \to (0,+\infty)$ be continuous functions such that f and $\frac{g}{f}$ are increasing. Prove that*

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x \le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t.$$

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Solution. First, we estimate the expression inside the integral sign on the left side of the given inequality. By the Chebycheff's inequality for integrals applied to increasing functions f and $\frac{g}{f}$ on the segment $[0,x]$ (where $x \in (0,1]$ is fixed), we get

$$\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)\left(\frac{1}{x}\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right) \le \frac{1}{x}\int_0^x g(t)\,\mathrm{d}t,$$

that is,

$$\frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \le \frac{x}{\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t} \tag{1}$$

for every $x \in (0,1]$. From the integral form of the Cauchy-Schwarz inequality on the segment $[0,x]$, we have

$$\left(\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)\left(\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t\right) \ge \left(\int_0^x t\,\mathrm{d}t\right)^2 = \frac{x^4}{4},$$

or

$$\frac{1}{\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t} \le \frac{4}{x^4}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t. \tag{2}$$

From (1) and (2) we obtain

$$\frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \le \frac{4}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t. \tag{3}$$

Finally, it remains to integrate (3) over $x \in (0,1]$ and to reverse the order of integration.

$$\begin{aligned}\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x &\le \int_0^1 \left(\int_0^x \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}t\right)\mathrm{d}x = \int_0^1 \left(\int_t^1 \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}x\right)\mathrm{d}t = \\ &= \int_0^1 \frac{4t^2 f(t)}{g(t)}\left(\int_t^1 \frac{\mathrm{d}x}{x^3}\right)\mathrm{d}t = \int_0^1 \frac{4t^2 f(t)}{g(t)}\left(\frac{1}{2t^2} - \frac{1}{2}\right)\mathrm{d}t = \\ &= 2\int_0^1 \frac{f(t)}{g(t)}(1-t^2)\,\mathrm{d}t \le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t.\end{aligned}$$

(*Remark.* The constant 2 on the right hand side of the given inequality is optimal, i.e., the least possible. Consider $f(t) := 1$ and $g(t) := t + \varepsilon$ for some fixed $\varepsilon > 0$. Then

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x = \int_0^1 \frac{x}{\frac{1}{2}x^2 + \varepsilon x}\,\mathrm{d}x = 2\int_0^1 \frac{\mathrm{d}x}{x + 2\varepsilon} = 2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon$$

and

$$\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{t+\varepsilon} = \ln(1+\varepsilon) - \ln\varepsilon.$$

The quotient of these two expressions can be made arbitrarily close to 2 since

$$\lim_{\varepsilon \searrow 0} \frac{2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon}{\ln(1+\varepsilon) - \ln\varepsilon} = 2\lim_{\varepsilon \searrow 0} \frac{-\frac{\ln(1+2\varepsilon)}{\ln\varepsilon} + \frac{\ln 2}{\ln\varepsilon} + 1}{-\frac{\ln(1+\varepsilon)}{\ln\varepsilon} + 1} = 2.$$

Therefore, the constant 2 is the best possible one.) □